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Uniqueness of symmetric basis in quasi-Banach spaces

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ABSTRACT

We show that if X is a nonlocally convex natural quasi-Banach space with symmetric basis whose Banach envelope is isomorphic to ℓ_1, then all symmetric bases of X are equivalent. The scope of this result is quite ample since the Banach envelopes of natural quasi-Banach spaces with basis always exhibit an ℓ_1-like behavior, in the sense that they contain copies of ℓ_1^n's uniformly complemented.

1. Introduction

A quasi-Banach space X is said to have a unique unconditional or symmetric basis if for any two normalized bases of the same type, there exists an automorphism of X which takes one basis to the other.

If X is locally convex (i.e., X is Banach space) then X has a unique unconditional basis if and only if it is isomorphic to one of the spaces ℓ_2, ℓ_1 or c_0 [17]. The class of Banach spaces with a unique symmetric basis is considerably larger and it contains all the Orlicz sequence spaces ℓ_F for which $\lim_{t\to 0} tF'(t)/F(t)$ exists [15], and also the Lorentz sequence spaces $d(w,p)$ where $p \geqslant 1$ and $w = (w_n)_{n=1}^{\infty} \in c_0 \setminus \ell_1$ is a nonincreasing sequence of positive numbers with $w_1 = 1$ [6]. However, a complete classification of Banach spaces with a unique symmetric basis seems far from being achieved.

In [8], Kalton investigated Orlicz sequence spaces when the restriction of local convexity is lifted and proved that if $\lim_{t\to 0} F(t)/t = \infty$ then l_F has a unique unconditional basis (hence, a unique symmetric basis). The ℓ_p spaces for $0 < p < 1$ belong to this class. Nawrocki and Ortyński [18] studied the quasi-Banach Lorentz sequence spaces $d(w,p)$ for $0 < p < 1$, described their Banach envelopes, and showed that all symmetric bases of $d(w,p)$ for $0 < p < 1$ are equivalent (see also [19]). The question of uniqueness of unconditional basis in $d(w,p)$ for $0 < p < 1$, on the other hand, was initiated in [18], continued in [12], and has been recently completely settled in [4].

All the above mentioned spaces have a 1-symmetric, and so 1-unconditional, basis which induces a p-convex lattice structure for some $p > 0$. If a quasi-Banach space X is isomorphic to a closed subspace of a p-convex quasi-Banach lattice, then X is also p-convex and it is called *natural* (see [9]).

Our aim is to prove that all natural quasi-Banach spaces with symmetric basis whose Banach envelope is isomorphic to ℓ_1 have a unique symmetric basis. This turns out to be the most important case since the Banach envelope of a natural quasi-Banach space with a basis contains ℓ_1^n's uniformly complemented [10]. Let us point out that if a quasi-Banach space is not natural then its Banach envelope can behave very differently, as shown in [7, Sections 6 & 7].

Throughout this article we use standard Banach space theory terminology and notation, as may be found in [1,16]. For the necessary background on quasi-Banach spaces we refer the reader to the books [11,20] and the paper [13].

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2. The main result

Suppose that $(x_n)_{n=1}^{\infty}$ is a symmetric basis with symmetric constant K of a quasi-Banach space $(X, \|\cdot\|)$ and assume that the Banach envelope of X, $(\hat{X}, \|\cdot\|_c)$ is isomorphic to $(\ell_1, \|\cdot\|_1)$. We observe that then $(x_n)_{n=1}^{\infty}$ is also a K-symmetric basis of $\hat{X}$ and that

$$K^{-1} = K^{-1}\|x_n\| \leqslant \|x_n\|_c \leqslant \|x_n\| = 1, \quad n \in \mathbb{N}.$$

That is, $(x_n)_{n=1}^{\infty}$ is a semi-normalized symmetric basis of $\hat{X}$ equivalent to the normalized basis $(x_n/\|x_n\|_c)_{n=1}^{\infty}$. Since ℓ_1 has unique symmetric basis, $(x_n)_{n=1}^{\infty}$ is equivalent to the canonical ℓ_1-basis, i.e., there is a constant $D > 0$ so that

$$D^{-1}\sum_{n=1}^{\infty}|\alpha_n| \leqslant \left\|\sum_{n=1}^{\infty}\alpha_n x_n\right\|_c \leqslant D\sum_{n=1}^{\infty}|\alpha_n|,$$

for any sequence of scalars $(\alpha_n)_{n=1}^{\infty} \in c_{00}$.

Our main result, Theorem 2.2, relies upon the following lemma. The technique of finding "uniformly large coefficients" was introduced by Kalton in [8] to prove the uniqueness of unconditional basis in nonlocally convex Orlicz sequence spaces and was extended to the framework of quasi-Banach lattices in [12]. It has become crucial to further determine the uniqueness of unconditional basis in other quasi-Banach spaces (cf. [2,3,12,14]).

Lemma 2.1. *(See [12, Theorem 2.3].) Let X be a natural quasi-Banach space with two normalized unconditional bases $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$. Let $(x_n^*)_{n=1}^{\infty}$ and $(y_n^*)_{n=1}^{\infty}$ be the sequences of biorthogonal linear functionals associated to $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$, respectively. Set $S \subset \mathbb{N}$ and suppose that there are a constant $\beta > 0$ and an injective map $\sigma: S \to \mathbb{N}$ so that*

$$\left|x_{\sigma(n)}^*(y_n)\right| \geqslant \beta, \quad n \in S.$$

Then, there is a constant $\rho > 0$ such that for any scalars $(a_n)_{n=1}^{\infty} \in c_{00}$,

$$\rho\left\|\sum_{n\in S}a_n x_{\sigma(n)}\right\| \leqslant \left\|\sum_{n\in S}a_n y_n\right\|.$$

Theorem 2.2. *Let X be a natural quasi-Banach space with normalized symmetric basis $(x_n)_{n=1}^{\infty}$ and whose Banach envelope is isomorphic to ℓ_1. Suppose that $(y_n)_{n=1}^{\infty}$ is another normalized symmetric basis of X. Then there exist positive constants Δ_1, Δ_2 so that*

$$\Delta_1\left\|\sum_{n=1}^{\infty}a_n x_n\right\| \leqslant \left\|\sum_{n=1}^{\infty}a_n y_n\right\| \leqslant \Delta_2\left\|\sum_{n=1}^{\infty}a_n x_n\right\|, \tag{2.1}$$

for any choice of scalars $(a_n)_{n=1}^{\infty} \in c_{00}$.

Proof. The bases $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are symmetric bases in $\hat{X}$, both equivalent to the canonical basis of ℓ_1. Let $D > 0$ such that

$$\frac{1}{D}\sum_{n=1}^{\infty}|\alpha_n| \leqslant \left\|\sum_{n=1}^{\infty}\alpha_n x_n\right\|_c \leqslant D\sum_{n=1}^{\infty}|\alpha_n|,$$

whenever $\sum_{n=1}^{\infty}\alpha_n x_n \in X$, and $C > 0$ so that

$$\frac{1}{C}\sum_{n=1}^{\infty}|\alpha_n| \leqslant \left\|\sum_{n=1}^{\infty}\alpha_n y_n\right\|_c \leqslant C\sum_{n=1}^{\infty}|\alpha_n|,$$

whenever $\sum_{n=1}^{\infty}\alpha_n y_n \in X$.

Put

$$x_k = \sum_{n=1}^{\infty}y_n^*(x_k)y_n, \quad k = 1, 2, \ldots,$$

and

$$y_n = \sum_{k=1}^{\infty}x_k^*(y_n)x_k, \quad n = 1, 2, \ldots.$$

For each fixed $k \in \mathbb{N}$ we have

$$1 = \left|x_k^*(x_k)\right| \leqslant \sum_{n=1}^{\infty}\left|y_n^*(x_k)x_k^*(y_n)\right| \leqslant \sup_n\left|x_k^*(y_n)\right|\sum_{n=1}^{\infty}\left|y_n^*(x_k)\right| \leqslant C\sup_n\left|x_k^*(y_n)\right|\left\|\sum_{n=1}^{\infty}y_n^*(x_k)y_n\right\|_c$$
$$= C\sup_n\left|x_k^*(y_n)\right|\|x_k\|_c \leqslant C\sup_n\left|x_k^*(y_n)\right|.$$

Thus, we can define a function $f:\mathbb{N}\to\mathbb{N}$ by $f(k)=n(k)$, where $n(k)$ is such that $|x_k^*(y_{n(k)})| > 1/2C$. Of course, f need not be one-to-one nor onto, but it is not far from being injective. Indeed, given $n_0 \in \mathbb{N}$ suppose there are integers $k_1,\dots,k_N$ so that

$$\left|x_{k_1}^*(y_{n_0})\right| > 1/2C, \quad \dots, \quad \left|x_{k_N}^*(y_{n_0})\right| > 1/2C.$$

Then,

$$\frac{N}{2C} \leqslant \sum_{k=1}^{\infty}\left|x_k^*(y_{n_0})\right| \leqslant D\left\|\sum_{k=1}^{\infty}x_k^*(y_{n_0})x_k\right\|_c = D\|y_{n_0}\|_c \leqslant D.$$

Hence there is a partition of $\mathbb{N}$ into at most $N \leqslant 2DC$ disjoint subsets $A_1,\dots,A_N$ so that the restriction of f to each A_i is one-to-one. We pick one of these sets of the same cardinality as $\mathbb{N}$, say A_1, and let $S=f(A_1)$. This way, the map $f:A_1\to S$ is a bijection. Let us consider its inverse,

$$\sigma : S\to A_1, \quad n\to\sigma(n) \quad \text{so that } \left|x_{\sigma(n)}^*(y_n)\right| > 1/2C.$$

By appealing to Lemma 2.1 we find $\rho>0$ so that for any $(a_n)_{n=1}^{\infty}\in c_{00}$ we have

$$\rho\left\|\sum_{n\in S}a_nx_{\sigma(n)}\right\| \leqslant \left\|\sum_{n\in S}a_ny_n\right\|. \tag{2.2}$$

The symmetry of the bases yields positive constants Γ_1, Γ_2 such that

$$\frac{1}{\Gamma_1}\left\|\sum_{n=1}^{\infty}a_nx_n\right\| \leqslant \left\|\sum_{n\in S}a_nx_{\sigma(n)}\right\| \leqslant \Gamma_1\left\|\sum_{n=1}^{\infty}a_nx_n\right\| \tag{2.3}$$

and

$$\frac{1}{\Gamma_2}\left\|\sum_{n=1}^{\infty}a_ny_n\right\| \leqslant \left\|\sum_{n\in S}a_ny_n\right\| \leqslant \Gamma_2\left\|\sum_{n=1}^{\infty}a_ny_n\right\|, \tag{2.4}$$

for any $(a_n)_{n=1}^{\infty}\in c_{00}$. Combining (2.2)–(2.4) we obtain the left-hand side inequality of Theorem 2.2 with $\Delta_1=(\Gamma_1\Gamma_2)^{-1}\rho$. The right-hand side inequality in (2.1) follows by interchanging the roles of the two bases by a symmetric argument, and this completes the proof. □

Concluding remarks. (a) The hypotheses of Theorem 2.2 are not sufficient to ensure the uniqueness of *unconditional* basis in X. As a matter of fact, Kalton [8] found examples of nonlocally convex Orlicz sequence spaces whose Banach envelope is ℓ_1 and, however, do not have a unique unconditional basis.

(b) There are natural quasi-Banach spaces whose Banach envelope is not ℓ_1 with unique symmetric basis. Take, for instance, $X=d(w,p)$ for $0<p<1$, where the sequence of weights $(w_n)_{n=1}^{\infty}$ satisfies the condition

$$\inf_n\frac{(w_1+\cdots+w_n)^{1/p}}{n}=0.$$

Then, by [18, Proposition 3], X has unique symmetric basis, but by [18, Theorem 1], the Banach envelope of X is not isomorphic to ℓ_1.

(c) A normalized basis is *perfectly homogeneous* if it is equivalent to all its normalized block basic sequences, that is, perfectly homogeneous bases are a special case of symmetric bases. For Banach spaces, a classical result of Zippin [21] proved that perfectly homogeneous bases are equivalent to either the canonical c_0-basis or the canonical ℓ_p-basis for some $1\leqslant p<\infty$. In [5] the authors show that Zippin's techniques can be extended to prove that perfectly homogeneous bases of nonlocally convex quasi-Banach spaces are always equivalent to the canonical ℓ_p-basis for some $0<p<1$.

Our work leaves open the following question.

Problem 2.3. Are there any nonlocally convex quasi-Banach spaces with symmetric basis without the property of uniqueness of symmetric basis?

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